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Integral Formulas for Non-Self-Adjoint Distributed Dynamic Systems

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Integral formulas for analytical prediction of the dynamic response of a class of non-self-adjoint distributed systems are studied. Combined non-self-adjoint effects of damping, gyroscopic, and circulatory forces are examined. The response of the distributed system subject to arbitrary external, boundary, and initial disturbances is obtained in a closed-form Green's function integral. The Green's function is expanded in an eigenfunction series, without assuming completeness of system eigenfunctions. In addition, a generalized reciprocal theorem is derived.

I. Introduction

DISTRIBUTED dynamic systems whose parameters are dependent on the spatial domain are common in many branches of science and technology. The mathematical description of these systems usually leads to boundary–initial value problems associated with partial differential equations. The development of modern technologies, including optimal design and active control of flexible structures, requires accurate prediction of the dynamic response of distributed systems. This work investigates integral formulas for analytical prediction of the dynamic response of a class of linear distributed systems that have combined non-self-adjoint effects of damping, gyroscopic, and circulatory forces, and are under arbitrary external, boundary, and initial disturbances. Mainly, three issues are addressed: 1) Green's function formulation for transient response prediction, 2) modal expansion of system Green's function, and 3) a generalized reciprocal principle. These issues are important to structural dynamics and structural control and lay a foundation for the development of analytical and numerical solution methods.

The concept of Green's functions is by no means new. It was first introduced by G. Green as early as 1828 and since has been applied to various problems in mathematical physics. Several excellent monographs on the subject have been published.[1–4] Instead of its many advantages, the Green's function method has been for formal and abstract mathematical analyses, and needs further development before becoming a practical tool for practical engineering problems. In the literature, most studies employing the Green's function method are focused on self-adjoint systems although some specific non-self-adjoint systems have been considered.[5,6] The Green's function formula for general non-self-adjoint distributed systems with damping, gyroscopic, and circulatory forces is not available.

The utility of the Green's function method in a specific problem depends on the determination of the Green's function. It is well known that the Green's function of a self-adjoint system can be expressed by its orthogonal eigenfunctions in an infinite series. Unfortunately, the classic modal analysis is not valid for non-self-adjoint systems whose eigenfunctions are nonorthogonal. Under the circumstances, modal expansion of the system response by biorthogonal state–space eigenfunctions is often pursued.[7,8] The state–space approach, although able to give an eigenfunction representation of the Green's function, has two problems. First, the approach assumes that the state–space eigenfunctions of a non-self-adjoint distributed system form a complete basis in an infinite-dimensional function space, which may or may not be true, and is difficult to justify. Consequently, the legitimacy and convergence of such state–space modal expansion is not ensured. Second, the approach has to estimate the adjoint state–space eigenfunctions whose physical meaning has not been well classified and therefore can be computationally expensive.

Recently, the author[9] examined a class of nonproportionally damped distributed systems. On the basis of the symmetric property of the damping operator and a direct inverse Laplace transform procedure, the closed-form transient response of the damped system was obtained in an eigenfunction series. The eigenfunction expansion, however, is not applicable to distributed dynamic systems with damping, gyroscopic, and circulatory forces, whose existence in general renders spacial differential operators asymmetric. Furthermore, the inverse Laplace transform procedure is based on a residue theorem, whose validity for general non-self-adjoint distributed systems remains to be proven.

This paper presents, for the first time, a modal expansion of the dynamic response of general non-self-adjoint distributed systems, without assuming completeness of system eigenfunctions. Extending the inverse Laplace transform concept,[9] the key in the analysis is to express the system Green's function in terms of the transfer function residues. In the development, a residue theorem for inverse Laplace transform is first proven. Through establishment of a relationship between the adjoint state–space eigenfunctions and the modes of vibration, the transfer function residues are related to the nonorthogonal eigenfunctions associated with the original equations of motion. As such, a convergent eigenfunction representation of the Green's function is obtained. In the meantime, because the adjoint state–space eigenfunctions are not needed in the evaluation, substantial computations can be saved.

As the last part of this investigation, a generalized reciprocal theorem is derived for general non-self-adjoint systems. In the existing reciprocal theorems for classic self-adjoint systems[10] and viscoelastic bodies,[11] the response of a continuum under one set of loads is expressed by that of the same continuum under another set of loads. In the new reciprocal theorem, the response of a distributed system is related to that of the corresponding auxiliary system. The auxiliary system in many cases is the same as the original one. The reciprocal relation also leads to the Green's function formula.

II. Green's Function Formula

The distributed dynamic system is governed by the partial differential equation

$$Mw_{,tt}(x,t) + (D+G)w_{,t}(x,t) + (K+N)w(x,t) = f(x,t), \qquad x \in \Omega \tag{1a}$$

with N_b boundary conditions

$$\Gamma_j w(x,t) = \gamma_j(x,t), \qquad x \in \partial\Omega, \quad j = 1, 2, \ldots, N_b \tag{1b}$$

and the initial conditions

$$w(x,0) = a_0(x), \qquad w_{,t}(x,0) = b_0(x), \qquad x \in \Omega \tag{1c}$$

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where $w(x,t)$ is the displacement of the distributed system; $f(x,t)$ is the external force; $\gamma_j(x,t)$ are the given boundary disturbances (either external loads or foundation motions); $a_0(x)$ and $b_0(x)$ are the initial displacement and velocity, respectively; $(\,)_{,t} = \partial(\,)/\partial t$; Ω is a bounded open region with boundary $\partial\Omega$; and Γ_j are the spatial boundary operators. The spatial differential operators M and K represent the inertia and stiffness of the distributed system, respectively, and D, G, and N are associated with the damping, gyroscopic, and circulatory forces, respectively.[12,13] M, D, and K are symmetric and G and N are skew-symmetric:

$$\langle Mu, v\rangle = \langle u, Mv\rangle, \qquad \langle Du, v\rangle = \langle u, Dv\rangle$$

$$\langle Ku, v\rangle = \langle u, Kv\rangle, \qquad \langle Gu, v\rangle = -\langle u, Gv\rangle$$

$$\langle Nu, v\rangle = -\langle u, Nv\rangle$$

where the inner product is defined by $\langle u, v\rangle = \int_\Omega \bar{u} v \, dx$ with u and v being comparison functions from a Hilbert space and $\bar{u}$ the complex conjugate of u. Also, M and K are positive definite and D is positive semidefinite. In this study, it is assumed that the boundary–initial problem is well posed and has a unique solution.

The fundamental problem is to solve the boundary–initial problem formed by Eqs. (1). To this end, a Green's function formula is derived by using Laplace transform. By superposition, the solution to Eqs. (1) is decomposed into

$$w(x,t) = y(x,t) + z(x,t) \tag{2}$$

where $y(x,t)$ satisfies

$$My_{,tt}(x,t) + (D+G)y_{,t}(x,t) + (K+N)y(x,t) = f(x,t), \qquad x \in \Omega \tag{3a}$$

$$\Gamma_j y(x,t) = 0, \qquad x \in \partial\Omega, \quad j = 1, 2, \ldots, N_b \tag{3b}$$

$$y(x,0) = a_0(x), \qquad y_{,t}(x,0) = b_0(x), \qquad x \in \Omega \tag{3c}$$

and $z(x,t)$ is the solution of

$$Mz_{,tt}(x,t) + (D+G)z_{,t}(x,t) + (K+N)z(x,t) = 0, \qquad x \in \Omega \tag{4a}$$

$$\Gamma_j z(x,t) = \gamma_j(x,t), \qquad x \in \partial\Omega, \quad j = 1, 2, \ldots, N_b \tag{4b}$$

$$z(x,0) = 0, \qquad z_{,t}(x,0) = 0, \qquad x \in \Omega \tag{4c}$$

Physically, $y(x,t)$ is the system response excited by the external and initial disturbances, and $z(x,t)$ by the boundary disturbances.

The integral formula for $y(x,t)$ is first derived. Laplace transform of Eqs. (3) with respect to time t gives

$$A(s)\hat{y}(x,s) = \hat{f}_{et}(x,s), \qquad x \in \Omega \tag{5a}$$

$$\Gamma_j \hat{y}(x,s) = 0, \qquad x \in \partial\Omega, \qquad j = 1, 2, \ldots, N_b \tag{5b}$$

where the circumflex ^ denotes Laplace transformation, the operator

$$A(s) = Ms^2 + (D+G)s + (K+N) \tag{6a}$$

and

$$f_{et}(x,s) \equiv \hat{f}(x,s) + M[sa_0(x) + b_0(x)] + (D+G)a_0(x), \qquad x \in \Omega \tag{6b}$$

The solution of Eqs. (6) is written in an integral form

$$\hat{y}(x,s) = \int_\Omega \hat{g}(x,\xi,s) f_{et}(\xi,s)\, d\xi, \qquad x \in \Omega \tag{7}$$

where the integral kernel $\hat{g}(x,\xi,s)$ is called the distributed transfer function.[1] Inverse Laplace transform of Eq. (7) and use of Eq. (6b) leads to

$$y(x,t) = \int_0^t \int_\Omega g(x,\xi,t-\tau) f(\xi,\tau)\, d\xi\, d\tau + \int_\Omega \{g_{,t}(x,\xi,t) Ma_0(\xi) + g(x,\xi,t) \times [Mb_0(\xi) + (D+G)a_0(\xi)]\}\, d\xi \tag{8}$$

where the Green's function $g(x,\xi,t)$ is the inverse Laplace transform of $\hat{g}(x,\xi,s)$. Thus, the response of the distributed dynamic system to both external and initial disturbances is represented by the Green's function integral, although the Green's function itself has yet to be determined.

The distributed transfer function is the solution of the differential equations[4]

$$A^T(s)\hat{g}(x,\xi,s) = \delta(x-\xi), \qquad \xi \in \Omega \tag{9a}$$

$$\Gamma_j^* \hat{g}(x,\xi,s) = 0, \qquad \xi \in \partial\Omega, \quad j = 1, 2, \ldots, N_b \tag{9b}$$

where

$$A^T(s) = Ms^2 + (D-G)s + (K-N) \tag{10}$$

and Γ_j^*, the adjoint operators to Γ_j, are obtained from

$$\int_\Omega \{A(s)[u(x)]v(x) - u(x)A^T(s)[v(x)]\}\, dx = \int_{\partial\Omega} \sum_{j=1}^{N_b} \{B_j[u(x)]\Gamma_j^* v(x) - \Gamma_j u(x) E_j[v(x)]\}\, dx \tag{11}$$

with u and v being any differentiable functions and the operators B_j and E_j arising from integral by part. Likewise, the Green's function is governed by

$$Mg_{,tt}(x,\xi,t) + (D-G)g_{,t}(x,\xi,t) + (K-N)g(x,\xi,t) = \delta(x-\xi)\delta(t), \qquad \xi \in \Omega \tag{12a}$$

$$\Gamma_j^* g(x,\xi,t) = 0, \qquad \xi \in \partial\Omega, \quad j = 1, 2, \ldots, N_b \tag{12b}$$

$$g(x,\xi,0) = 0, \qquad g_{,t}(x,\xi,0) = 0, \qquad \xi \in \Omega \tag{12c}$$

In both Eqs. (9) and (12), all operators act on ξ. Apparently, Laplace transform of Eqs. (12) yields Eqs. (9).

Next, the solution form for $z(x,t)$ is derived. Laplace transform of Eqs. (4) gives

$$A(s)\hat{z}(x,s) = 0, \qquad x \in \Omega \tag{13a}$$

$$\Gamma_j \hat{z}(x,s) = \hat{\gamma}_j(x,s), \qquad x \in \partial\Omega, \quad j = 1, 2, \ldots, N_b \tag{13b}$$

where $\hat{\gamma}_j(x,s)$ are the Laplace transforms of the boundary disturbances $\gamma_j(x,t)$. With Eqs. (11) and (13), consider the integral

$$\int_\Omega A(s)[\hat{z}(\xi,s)]\hat{g}(x,\xi,s)\, d\xi = \int_\Omega \hat{z}(\xi,s) A^T(s)[\hat{g}(x,\xi,s)]\, d\xi + \int_{\partial\Omega} \sum_{j=1}^{N_b} \{\Gamma_j^* \hat{g}(x,\xi,s) B_j[\hat{z}(\xi,s)] - E_j[\hat{g}(x,\xi,s)]\Gamma_j \hat{z}(\xi,s)\}\, d\xi = 0$$

By Eq. (9a) and the boundary conditions (4b) and (9b), the above becomes

$$\hat{z}(x,s) = \int_{\partial\Omega} \sum_{j=1}^{N_b} E_j[\hat{g}(x,\xi,s)]\hat{\gamma}_j(\xi,s)\, d\xi \tag{14}$$

Inverse Laplace transform of Eq. (14) yields

$$z(x,t)=\int_0^t\int_{\partial\Omega}\sum_{j=1}^{N_b}h_j(x,\xi,t-\tau)\gamma_j(\xi,\tau)\,d\xi\,d\tau \tag{15}$$

where the boundary influence functions

$$h_j(x,\xi,t)=\mathcal{L}^{-1}\{E_j[\hat{g}(x,\xi,s)]\},\qquad j=1,2,\ldots,N_b \tag{16}$$

and $\mathcal{L}^{-1}$ is the inverse Laplace transform operator. The operators E_j generally have the form

$$E_j=E_{1j}s+E_{0j} \tag{17}$$

where E_{ij} are spatial operators. The $E_{1j}s$ usually appear when operators such as $d\partial^5/\partial x^4\partial t$, which models material damping in a beam, are involved in the equations of motion. By Eq. (17), the boundary influence functions

$$h_j(x,\xi,t)=(E_{1j}\partial/\partial t+E_{0j})g(x,\xi,t) \tag{18}$$

Finally, substitute Eqs. (8), (15), and (18) into Eq. (2) to obtain an integral formula for complete solution of the original boundary–initial value problem described by Eqs. (1):

$$\begin{aligned}w(x,t)&=\int_0^t\int_\Omega g(x,\xi,t-\tau)f(\xi,\tau)\,d\xi\,d\tau\\&+\int_0^t\int_{\partial\Omega}\sum_{j=1}^{N_b}(E_{1j}\partial/\partial t+E_{0j})\\&\times[g(x,\xi,t-\tau)]\gamma_j(\xi,\tau)\,d\xi\,d\tau+\int_\Omega\{g_{,t}(x,\xi,t)Ma_0(\xi)\\&+g(x,\xi,t)[Mb_0(\xi)+(D+G)a_0(\xi)]\}\,d\xi\end{aligned} \tag{19}$$

The dynamic response of the distributed system is attributable to external, boundary, and initial disturbances. Equation (19) shows that the contributions of each type of disturbances can be expressed by the system Green's function. In other words, the key to solving Eqs. (1) is to determine the Green's function from Eqs. (12); the effects of the boundary and initial disturbances can be added easily once the Green's function is known.

It should be mentioned that the Green's function formula is also obtainable more vigorously by the semigroup theory. Indeed, the way to construct the solution to Eqs. (1) is not unique. The main reason for this investigation to adopt the Laplace transform approach is to provide an analysis that might be more easily understood and utilized by engineers.

III. Eigenvalue Problems

The eigenvalue problems of the non-self-adjoint distributed system are investigated, to prepare for the development of a new modal expansion of the Green's function and transient response in Sec. IV. The eigenvalue problem associated with Eqs. (1) is

$$\{\lambda_k^2M+\lambda_k(D+G)+(K+N)\}u_k(x)=0,\qquad k=\pm1,\pm2,\ldots \tag{20}$$

where $\lambda_k\in C$ and $u_k(x)$ are the kth eigenvalue and eigenfunction (mode shape) of the distributed system. Without loss of generality, assume that the eigenvalues are distinct. This assumption, however, can be removed by using the Schmidt orthogonalization procedure.[3] Also, assume that the distributed system has no rigid-body modes; i.e., $|\lambda_k|\neq0$, for any k. The adjoint eigenvalue problem to Eq. (20) is defined as

$$\{\bar{\lambda}_k^2M+\bar{\lambda}_k(D-G)+(K-N)\}v_k(x)=0,\qquad k=\pm1,\pm2,\ldots \tag{21}$$

the overbar denotes complex conjugation. The eigensolutions have the properties $\lambda_{-k}=\bar{\lambda}_k$, $u_{-k}=\bar{u}_k$, and $v_{-k}=\bar{v}_k$. Except for classic self-adjoint systems and proportionally damped systems, the eigenvectors u_k and v_k are complex and nonorthogonal.

Equivalent eigenvalue problems in a state space are also considered. Equation (1a) is transformed into[7]

$$A_0\eta_{,t}(x,t)=A_1\eta(x,t)+q(x,t) \tag{22}$$

where

$$\eta=\begin{pmatrix}w\\w_{,t}\end{pmatrix},\qquad A_0=\begin{bmatrix}1&0\\0&M\end{bmatrix}$$
$$A_1=\begin{bmatrix}0&1\\-K-N&-D-G\end{bmatrix},\qquad q=\begin{pmatrix}0\\f\end{pmatrix} \tag{23}$$

The associate and adjoint eigenvalue problems are

$$A_1\phi_k(x)=\lambda_kA_0\phi_k(x) \tag{24}$$

$$A_1^T\psi_k(x)=\bar{\lambda}_kA_0\psi_k(x) \tag{25}$$

where $k=\pm1,\pm2,\ldots$, and $\phi_k(x)$ are related to the mode shapes $u_k(x)$ by

$$\phi_k(x)=\begin{pmatrix}u_k(x)\\\lambda_ku_k(x)\end{pmatrix} \tag{26}$$

The state–space eigenfunctions are in the biorthogonality relations

$$[\psi_j,A_0\phi_k]=[A_0\psi_j,\phi_k]=2\delta_{jk} \tag{27a}$$

$$[\psi_j,A_1\phi_k]=[A_1^T\psi_j,\phi_k]=2\lambda_j\delta_{jk} \tag{27b}$$

where the inner product is defined by

$$[a,b]=\langle a_1,a_2\rangle+\langle b_1+b_2\rangle=\int_\Omega[\bar{a}_1(x)b_1(x)+\bar{a}_2(x)b_2(x)]\,dx$$

for $a=(a_1,a_2)^T$ and $b=(b_1,b_2)^T$. Following Ref. 14, it can be shown that the adjoint eigenfunctions of problems (21) and (25) are related by

$$\psi_k(x)=\frac{1}{\bar{\lambda}_k^2}\begin{pmatrix}-(K-N)v_k(x)\\\bar{\lambda}_kv_k(x)\end{pmatrix},\qquad k=\pm1,\pm2,\ldots \tag{28}$$

The normalization condition for ϕ_k and ψ_k, by Eqs. (20), (27), and (28), is written as

$$\begin{aligned}[\psi_k,A_0\phi_k]&=\int_\Omega\bar{v}_kMu_k\,dx-\frac{1}{\lambda_k^2}\int_\Omega\bar{v}_k(K+N)u_k\,dx\\&=2\int_\Omega\bar{v}_kMu_k\,dx+\frac{1}{\lambda_k}\int_\Omega\bar{v}_k(D+G)u_k\,dx=2\end{aligned} \tag{29}$$

While state–space formulism has been widely used, the physical meaning of adjoint eigenfunctions of distributed systems with combined non-self-adjoint effects of damping, gyroscopic, and circulatory forces is not clear. Extending a recent study on eigensolutions of lumped parameter systems,[14] the following discussion shows that the adjoint eigenfunctions, $v_k(x)$ and therefore $\psi_k(x)$, relate to the mode shapes of vibration of the system. In the interest of simplicity, it is assumed that the boundary operators Γ_j and Γ_j^* are the same, which is true for many physical systems. [If Γ_j and Γ_j^* are not identical, $v_k(x)$ are the mode shapes of an auxiliary system, as defined in Sec. V.]

a) Classic self-adjoint systems ($D=G=N=0$):

$$Mw_{,tt}(x,t)+Kw(x,t)=f(x,t),\qquad x\in\Omega \tag{30}$$

Being self-adjoint, the system has purely imaginary eigenvalues, and real and orthogonal eigenfunctions.[15] In this case, the eigenvalue problems (20) and (21) are identical, and $v_k(x)$ are the same as the mode shapes $u_k(x)$.

b) Undamped gyroscopic systems ($D=N=0$):

$$\begin{aligned}&Mw_{,tt}(x,t)+Gw_{,t}(x,t)\\&\quad+Kw(x,t)=f(x,t),\qquad x\in\Omega\end{aligned} \tag{31}$$

The associated eigenvalue problems are

$$\{\lambda_k^2 M + \lambda_k G + K\}u_k = 0 \tag{32a}$$

$$\{\bar{\lambda}_k^2 M - \bar{\lambda}_k G + K\}v_k = 0 \tag{32b}$$

The eigenvalues are purely imaginary, and the eigenvectors are complex and nonorthogonal.[13] Accordingly, $\bar{\lambda}_k = -\lambda_k$, and Eq. (32b) becomes

$$\{\lambda_k^2 M + \lambda_k G + K\}v_k = 0 \tag{33}$$

Comparing Eqs. (32a) and (33) gives

$$v_k(x) = u_k(x) \tag{34}$$

which means that $v_k(x)$ describe the mode shape distributions of the gyroscopic system. A similar result has been obtained.[6]

c) Damped nongyroscopic systems ($G = N = 0$):

$$Mw_{,tt}(x,t) + Dw_{,t}(x,t) + Kw(x,t) = f(x,t), \qquad x \in \Omega \tag{35}$$

The associated eigenvalue problems are

$$\{\lambda_k^2 M + \lambda_k D + K\}u_k = 0 \tag{36a}$$

$$\{\bar{\lambda}_k^2 M + \bar{\lambda}_k D + K\}v_k = 0 \tag{36b}$$

Complex conjugation of Eq. (36b) and comparison with Eq. (36a) leads to

$$v_k(x) = \bar{u}_k(x) = u_{-k}(x) \tag{37}$$

The adjoint eigenvectors $v_k(x)$ are the mode shapes, whether or not the distributed system is proportionally damped.

d) Damped gyroscopic systems ($N = 0$):

$$Mw_{,tt}(x,t) + (D+G)w_{,t}(x,t) + Kw(x,t) = f(x,t), \qquad x \in \Omega \tag{38}$$

In many cases, the gyroscopic and stiffness operators for system (38) are dependent on a physical parameter α, with the properties

$$G = G(\alpha) = -G(-\alpha), \qquad K = K(\alpha) = K(-\alpha) \tag{39}$$

Examples of such systems include band saws and circular saws, computer tape- or disk-drive systems, rotating shafts and blades, rotating cylindrical shells, power transmission chains and belts, pipes conveying fluid, and paper webs. The parameter α in these systems is either the transport velocity of axially moving continua or the rotation speed of rotating bodies. Define an auxiliary system

$$M\vartheta_{,tt}(x,t) + [D + G(-\alpha)]\vartheta_{,t}(x,t) + K(-\alpha)\vartheta(x,t) = f(x,t), \qquad x \in \Omega \tag{40}$$

which actually is the same as system (38) but with its transport velocity or rotation speed in the opposite direction. The adjoint eigenvalue problem of the original system (38) is

$$\{\bar{\lambda}_k^2 M + \bar{\lambda}_k (D - G) + K\}v_k = 0 \tag{41}$$

The eigenvalue problem of the auxiliary system, on the other hand, is

$$\{\mu_k^2 M + \mu_k (D - G) + K\}\theta_k = 0 \tag{42}$$

where μ_k and $\theta_k(x)$ are the eigensolutions. It is seen that the solutions of the eigenvalue problems (41) and (42) are in the relationships

$$\mu_k = \bar{\lambda}_k, \qquad \theta_k(x) = v_k(x) \tag{43}$$

Therefore, the adjoint eigenvectors $v_k(x)$ are the mode shapes of the auxiliary system (40). A similar analysis is applicable to those distributed systems whose circulatory operator has the property $N = N(\beta) = -N(-\beta)$, where β is a physical parameter such as an axial load applied to a beam-column.

IV. Modal Expansion Theorem

The utility of the Green's function formula (19) depends on the knowledge of $g(x, \xi, t)$. If ϕ_k and ψ_k form a complete basis in the function space, the Green's function can be obtained in an infinite eigenfunction series. This is done by writing the state–space vector as

$$\eta(x,t) = \begin{pmatrix} w(x,t) \\ w_{,t}(x,t) \end{pmatrix} = \sum_{k=\pm 1}^{\pm\infty} \alpha_k(t)\phi_k(x) \tag{44}$$

where α_k are modal coordinates. Assuming zero boundary disturbances, substituting the above expression into Eq. (22) and applying the biorthogonality relations (27) yields an infinite number of decoupled differential equations for α_k. Solution of these equations and use of Eqs. (26), (28), and (44) leads to Eq. (8), with the Green's function given by

$$g(x,\xi,t) = \frac{1}{2}\sum_{k=\pm 1}^{\pm\infty} \frac{1}{\lambda_k} e^{\lambda_k t} u_k(x)\bar{v}_k(\xi), \qquad x, \xi \in \Omega \tag{45}$$

Although the above approach is straightforward, the completeness of the state–space eigenfunctions is difficult to justify, and may not be true. As a result, the legitimacy and convergence of the modal expansion (45) is not known. In this section, a different derivation based on inverse Laplace transform will verify the legitimacy and convergence of the modal expansion, without assuming completeness of the state–space eigenfunctions.

Let all of the eigenvalues, determined from Eq. (20), be located in the left part of the complex plane $Re(s) < \sigma$, $\sigma < +\infty$. Because $g(x, \xi, t)$ is the inverse Laplace transform of $\hat{g}(x, \xi, s)$,

$$g(x,\xi,t) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} e^{st}\hat{g}(x,\xi,s)\,\mathrm{d}s, \qquad i \equiv \sqrt{-1} \tag{46}$$

It is well known that the poles (singularities) of $\hat{g}(x, \xi, s)$ are the eigenvalues of the system. In the subsequent analysis, the Green's function is determined through evaluation of the complex integral (46) by a residue theorem.

Lemma. For $s = Re^{i\theta}$, there exists a constant $M_0 > 0$ such that

$$|\hat{g}(x,\xi,s)| \le M_0/R, \quad \text{as} \quad R \to \infty \tag{47}$$

Proof. By the initial value theorem[3] and Eq. (12c), $\lim_{s\to\infty} s\hat{g}(x,\xi,s) = \lim_{t\to 0} g(x,\xi,t) = 0$, indicating that for any $M_0 > 0$, $|s\hat{g}(x,\xi,s)|_{s=Re^{i\theta}} \le M_0$ for a large R. So, $|\hat{g}(x,\xi,s)| \le |s\hat{g}(x,\xi,s)|/|s| \le M_0/R$, for $s = Re^{i\theta}$ and $R \to \infty$.

Theorem 1. The Green's function of the distributed system, Eqs. (1), is expressed by

$$g(x,\xi,t) = \sum_{j=\pm 1}^{\pm\infty} e^{\lambda_j t} \operatorname*{Res}_{s=\lambda_j}\{\hat{g}(x,\xi,s)\} \tag{48}$$

where λ_j are the eigenvalues of the distributed system.

Proof. In the complex plane shown in Fig. 1, a Bromwich contour Σ consists of a straight line $AB = \{s \in C: Re(s) = \sigma, -R < Im(s) < R\}$ and an arc $\Pi_R = \{s \in C: s = Re^{i\theta}, -\theta_0 \le \theta \le \theta_0$. Here $\theta_0 = \cos^{-1}(\sigma/R)\}$, and $\sigma > Re(\lambda_k)$ for all k. As the radius

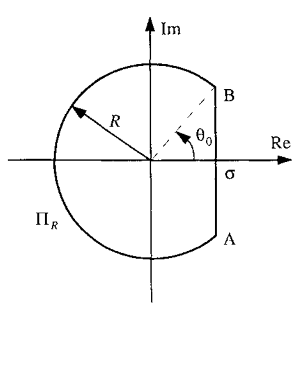

Fig. 1 Bromwich contour Σ consisting of line AB and arc Π_R.

$R \to \infty$, all of the poles of $\hat{g}(x, \xi, s)$ will be encircled by the contour, and the complex integral (46) becomes

$$g(x, \xi, t) = \frac{1}{2\pi i} \lim_{R \to \infty} \left\{ \int_{\Sigma} e^{st} \hat{g}(x, \xi, s)\,\mathrm{d}s - \int_{\Pi_R} e^{st} \hat{g}(x, \xi, s)\,\mathrm{d}s \right\} = \sum_{j=\pm 1}^{\pm\infty} e^{\lambda_j t} \underset{s=\lambda_j}{Res}\{\hat{g}(x, \xi, s)\} - \frac{1}{2\pi i} \lim_{R \to \infty} \int_{\Pi_R} e^{st} \hat{g}(x, \xi, s)\,\mathrm{d}s \tag{49}$$

Consider the case of $\sigma > 0$. Along the arc Π_R, the integral

$$\int_{\Pi_R} e^{st} \hat{g}(x, \xi, s)\,\mathrm{d}s = I_1 + I_2 + I_3 + I_4 \equiv \int_{\theta_0}^{\pi/2} + \int_{\pi/2}^{\pi} + \int_{\pi}^{3\pi/2} + \int_{3\pi/2}^{2\pi-\theta_0} \exp[tR(\cos\theta + i\sin\theta)] \times \hat{g}(x, \xi, Re^{i\theta})\,\mathrm{d}(Re^{i\theta})$$

Note that $R\cos\theta \le R\cos\theta_0 = \sigma$ for $\theta_0 \le \theta \le \pi/2$; see Fig. 1. By the Lemma,

$$|I_1| \le \int_{\theta_0}^{\pi/2} e^{\sigma t} \frac{M_0}{R} R\,\mathrm{d}\theta = M_0 e^{\sigma t}[\pi/2 - \cos^{-1}(\sigma/R)]$$

It can be shown that $\sin\psi \ge 2\psi/\pi$ for $0 \le \psi \le \pi/2$. So, by letting $\theta = \psi + \pi/2$,

$$|I_2| \le M_0 \int_{\pi/2}^{\pi} e^{tR\cos\theta}\,\mathrm{d}\theta = M_0 \int_0^{\pi/2} e^{-tR\sin\psi}\,\mathrm{d}\psi \le M_0 \int_0^{\pi/2} \exp[-2tR\psi/\pi]\,\mathrm{d}\psi = \frac{\pi}{2tR} M_0 (1 - e^{-tR})$$

Likewise,

$$|I_3| \le (\pi/2tR) M_0 (1 - e^{-tR})$$

and

$$|I_4| \le M_0 e^{\sigma t}[\pi/2 - \cos^{-1}(\sigma/R)]$$

As $R \to \infty$, all the $I_j (1 \le j \le 4)$ vanish, reducing Eq. (49) to Eq. (48). (The case of $\sigma \le 0$ can be similarly proven and is therefore omitted.)

Theorem 2. The residues of the distributed transfer function $\hat{g}(x, \xi, s)$ are of the form

$$\underset{s=\lambda_k}{Res}\{\hat{g}(x, \xi, s)\} = \frac{1}{\lambda_k[\psi_k, A_0\phi_k]} u_k(x)\bar{v}_k(x), \qquad k = \pm 1, \pm 2, \ldots \tag{50}$$

where $u_k(k)$ and $v_k(x)$ are the eigenfunctions determined from Eqs. (20) and (21).

Proof. For s near the pole λ_k, the distributed transfer function can be written as

$$\hat{g}(x, \xi, s) = [1/(s - \lambda_k)] b_k(x, \xi) + R(x, \xi, s) \tag{51}$$

where $b_k(x, \xi)$ is the residue of the transfer function at λ_k, and the function $R(x, \xi, s)$ is analytical at λ_k. Substituting Eq. (51) into Eq. (9a) gives

$$[1/(s - \lambda_k)] A^T(s)[b_k(x, \xi)] + A^T(s)[R(x, \xi, s)] = \delta(x - \xi)$$

where the operator $A^T(s)$, defined in Eq. (10), acts on ξ. The Cauchy integral

$$\frac{1}{2\pi i} \int_{\varepsilon_k} \left\{ \frac{1}{s - \lambda_k} A^T(s)[b_k(x, \xi)] + A^T(s)[R(x, \xi, s)] \right\} \mathrm{d}s = \frac{1}{2\pi i} \int_{\varepsilon_k} \delta(x - \xi)\,\mathrm{d}s$$

on the contour $\varepsilon_k = \{s \in C : |s - \lambda_k| = \varepsilon, \varepsilon < |\lambda_j - \lambda_k|$ for all $\lambda_j \ne \lambda_k\}$ yields

$$\left[M\lambda_k^2 + (D - G)\lambda_k + (K - N)\right] b_k(x, \xi) = 0, \qquad x, \xi \in \Omega \tag{52a}$$

Substituting Eq. (51) into the boundary condition (9b) and applying the Cauchy integral leads to

$$\Gamma_j^* b_k(x, \xi) = 0, \qquad x, \xi \in \partial\Omega \tag{52b}$$

It follows from Eqs. (52) that

$$b_k(x, \xi) = a(x)\bar{v}_k(\xi) \tag{53}$$

where $a(x)$ is a function of x. On the other hand, the transfer function also satisfies[4]

$$A(s)\hat{g}(x, \xi, s) = \delta(x - \xi), \qquad x, \xi \in \Omega \tag{54a}$$

$$\Gamma_j \hat{g}(x, \xi, s) = 0, \qquad x, \xi \in \partial\Omega \tag{54b}$$

where all of the operators act on x. Substituting Eqs. (51) and (53) into Eq. (54) and conducting similar Cauchy integration leads to

$$\left[M\lambda_k^2 + (D + G)\lambda_k + (K + N)\right] a(x) = 0, \qquad x \in \Omega \tag{55a}$$

$$\Gamma_j a(x) = 0, \qquad x \in \partial\Omega \tag{55b}$$

This implies that $a(x) = \beta_k u_k(x)$, and

$$b_k(x, \xi) = \beta_k u_k(x)\bar{v}_k(x) \tag{56}$$

where β_k is a nonzero constant.

To evaluate β_k, consider Eqs. (1) with zero boundary and initial disturbances. By Eqs. (19) and (12c), the displacement and velocity of the distributed system are

$$w(x, t) = \int_0^t \int_{\Omega} g(x, \xi, t - \tau) f(\xi, \tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \tag{57a}$$

$$w_{,t}(x, t) = \int_0^t \int_{\Omega} g_{,t}(x, \xi, t - \tau) f(\xi, \tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \tag{57b}$$

By Theorem 1 and Eqs. (56) and (57), the state–space vector is expressed by

$$\eta(x, t) = \begin{pmatrix} w(x, t) \\ w_{,t}(x, t) \end{pmatrix} = \sum_{k=\pm 1}^{\pm\infty} \beta_k \phi_k(x) \int_0^t \exp[\lambda_k(t - \tau)] f_k(\tau)\,\mathrm{d}\tau \tag{58}$$

where $f_k(t) = \int_{\Omega} u_k(\xi) f(\xi, t)\,\mathrm{d}\xi$. Substituting Eq. (58) into the state–space equation (22) and conducting Laplace transform yields

$$\sum_{k=\pm 1}^{\pm\infty} \beta_k \hat{f}_k(s) A_0 \phi_k(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \hat{f}(x, s)$$

where $\hat{f}_k(s)$ and $\hat{f}(x, s)$ are the Laplace transforms of $f_k(t)$ and $f(x, t)$, respectively. Take the inner product $[\psi_j, \cdot]$ of the above equation and apply the biorthogonality relation (27a) to obtain

$$\beta_k[\psi_k, A_0\phi_k] = 1/\lambda_k \tag{59}$$

Substituting Eq. (59) into Eq. (56) gives Eq. (50).

Corollary.

$$[\psi_k, A_0\phi_k] \ne 0 \quad \text{for all} \quad k \tag{60}$$

Condition (60) is taken for granted in many state–space modal analyses but has never been proven. As a byproduct from this investigation, condition (60) can be verified easily from Eqs. (59) and (56), where β_k should be finite to make sense.

Although the inner product $[\psi_k, A_0\phi_k]$ can take different values, depending on how the state–space eigenfunctions are scaled, it is easy to show that the residues

$$\frac{1}{\lambda_k[\psi_k, A_0\phi_k]} u_k(x)\bar{v}_k(x)$$

are invariant under any eigenfunction normalization. Thus, combining Theorems 1 and 2, the following modal expansion theorem is in order.

Theorem 3. The Green's function of the non-self-adjoint distributed system described by Eqs. (1) is uniquely determined by the convergent eigenfunction series

$$g(x,\xi,t)=\sum_{k=\pm1}^{\pm\infty} e^{\lambda_k t}\frac{1}{\lambda_k[\psi_k, A_0\phi_k]}u_k(x)\bar{v}_k(\xi) \tag{61}$$

where $[\psi_k, A_0\phi_k]$ can be any nonzero constants.

Without assuming completeness of the state–space eigenfunctions, it has been shown that the modal expansion of the Green's function of the non-self-adjoint distributed system is legitimate. As a special case, if the eigenfunction normalization (29) is used, Eq. (61) is identical to Eq. (45).

The modal expansion theorem lays a foundation for developing series solution methods for non-self-adjoint distributed systems. For instance, through substitution of Eq. (61) into Eq. (19), the solution to the boundary–initial value problem described in Eqs. (1) is given by

$$w(x,t)=\sum_{k=\pm1}^{\pm\infty}\frac{1}{\sigma_k}u_k(x)q_k(t) \tag{62}$$

where the time-dependent coordinates and normalization coefficients are

$$q_k(t)=e^{\lambda_k t}\int_0^t\left[\int_\Omega \bar{v}_k(\xi)f(\xi,\tau)\,d\xi + \int_{\partial\Omega}\sum_{j=1}^{N_b}(\lambda_k E_{1j}+E_{0j})\bar{v}_k(\xi)\gamma_j(\xi,\tau)\,d\xi\right]e^{-\lambda_k\tau}\,d\tau + e^{\lambda_k t}\int_\Omega \bar{v}_k(\xi)[(\lambda_k M+D+G)a_0(\xi)+Mb_0(\xi)]\,d\xi \tag{63a}$$

$$\sigma_k=\lambda_k[\psi_k, A_0\phi_k]=\int_\Omega \bar{v}_k(x)[2\lambda_k M+D+G]u_k(x)\,dx \tag{63b}$$

V. Generalized Reciprocal Theorem

Consider the following two distributed systems.

Original system:

$$\left\{M\frac{\partial^2}{\partial t^2}+(D+G)\frac{\partial}{\partial x}+(K+N)\right\}w_1(x,t)=f_1(x,t), \qquad x\in\Omega \tag{64a}$$

$$\Gamma_j w_1(x,t)=\gamma_{1j}(x,t), \qquad x\in\partial\Omega, \quad j=1,2,\ldots,N_b \tag{64b}$$

$$w_1(x,0)=a_1(x), \qquad w_{1,t}(x,0)=b_1(x), \qquad x\in\Omega \tag{64c}$$

Auxiliary system:

$$\left\{M\frac{\partial^2}{\partial t^2}+(D-G)\frac{\partial}{\partial x}+(K-N)\right\}w_2(x,t)=f_2(x,t), \qquad x\in\Omega \tag{65a}$$

$$\Gamma_j^* w_2(x,t)=\gamma_{2j}(x,t), \qquad x\in\partial\Omega, \quad j=1,2,\ldots,N_b \tag{65b}$$

$$w_2(x,0)=a_2(x), \qquad w_{2,t}(x,0)=b_2(x), \qquad x\in\Omega \tag{65c}$$

As discussed in Sec. III, in many cases, the auxiliary system (65) is or relates to the original system (64). Laplace transform of Eqs. (64) and (65) gives

$$A(s)\hat{w}_1(x,s)=q_1(x,s)\equiv\hat{f}_1(x,s)+M[sa_1(x)+b_1(x)] + (D+G)a_1(x), \qquad x\in\Omega \tag{66a}$$

$$\Gamma_j\hat{w}_1(x,s)=\hat{\gamma}_{1j}(x,s), \qquad x\in\partial\Omega, \quad j=1,2,\ldots,N_b \tag{66b}$$

and

$$A^T(s)\hat{w}_2(x,s)=q_2(x,s)\equiv\hat{f}_2(x,s)+M[sa_2(x)+b_2(x)] + (D-G)a_2(x), \qquad x\in\Omega \tag{67a}$$

$$\Gamma_j^*\hat{w}_2(x,s)=\hat{\gamma}_{2j}(x,s), \qquad x\in\partial\Omega, \quad j=1,2,\ldots,N_b \tag{67b}$$

According to Eq. (11), the integral

$$\begin{aligned}\int_\Omega\{\hat{w}_2(x,s)q_1(x,s)-\hat{w}_1(x,s)q_2(x,s)\}\,dx &= \int_\Omega\left\{\hat{w}_2(x,s)A(s)\hat{w}_1(x,s)-\hat{w}_1(x,s)A^T(s)\hat{w}_2(x,s)\right\}dx\\ &= \int_{\partial\Omega}\sum_{j=1}^{N_b}\{B_j[\hat{w}_1(x,s)]\hat{\gamma}_{2j}(x,s) - E_j[\hat{w}_2(x,s)]\hat{\gamma}_{1j}(x,s)\}\,dx\end{aligned} \tag{68}$$

The operators B_j and E_j arise from the integral by part and in general have the form

$$B_j=B_{1j}s+B_{0j}, \; E_j=E_{1j}s+E_{0j}$$

where B_{ij} and E_{ij} are spatial operators. Inverse Laplace transform of Eq. (68) leads to the following theorem.

Theorem 4. For the systems defined in Eqs. (64) and (65), their responses are in the reciprocal relation

$$\begin{aligned}&\int_0^t\int_\Omega f_2(x,\tau)w_1(x,t-\tau)\,dx\,d\tau + \int_0^t\int_{\partial\Omega}\sum_{j=1}^{N_b}\gamma_{2j}(\xi,\tau)\left(B_{1j}\frac{\partial}{\partial t}+B_{0j}\right)w_1(x,t-\tau)\,dx\,d\tau\\ &\quad+\int_\Omega\{w_{1,t}(x,t)Ma_2(x)+w_1(x,t)[Mb_2(x)+(D-G)a_2(x)]\}\,dx\\ &= \int_0^t\int_\Omega f_1(x,\tau)w_2(x,t-\tau)\,dx\,d\tau + \int_0^t\int_{\partial\Omega}\sum_{j=1}^{N_b}\gamma_{1j}(\xi,\tau)\left(E_{1j}\frac{\partial}{\partial t}t+E_{0j}\right)w_2(x,t-\tau)\,dx\,d\tau\\ &\quad+\int_\Omega\{w_{2,t}(x,t)Ma_1(x)+w_2(x,t)[Mb_1(x)+(D+G)a_1(x)]\}\,dx\end{aligned} \tag{69}$$

According to the reciprocal relation, the solution of the original system (64) can be determined from that of the auxiliary system (65). Also, note that Eq. (69) reduces to the Green's function formula (19) if only the impulse $f_2(x,t)=\delta(x-\xi)\delta(t)$ is applied to the auxiliary system (65).

VI. Example

For a simply supported translating beam shown in Fig. 2a, its dimensionless transverse displacement $w_1(x,t)$ is described by[16]

$$\left\{\frac{\partial^2}{\partial t^2}+\left(2c\frac{\partial}{\partial x}+d\right)\frac{\partial}{\partial t}+\left(c^2-T_0\right)\frac{\partial^2}{\partial x^2}+\frac{\partial^4}{\partial x^4}\right\}w_1(x,t) = f_1(x,t), \qquad x\in(0,1) \tag{70a}$$

$$\begin{aligned}&w_1(0,t)=y_{L1}(t), && \frac{\partial}{\partial x^2}w_1(0,t)=0\\ &w_1(1,t)=0, && \frac{\partial^2}{\partial x^2}w_1(1,t)=\tau_{R1}(t)\end{aligned} \tag{70b}$$

where c and T_0 are the transport speed and tension of the beam, d is the viscous damping coefficient, and $y_{L1}(T)$ and $\tau_{R1}(t)$ are the specified displacement and torque at the boundaries of the beam, respectively. The translating beam is a damped gyroscopic system, Eq. (38), with the spatial differential operators given by

$$M = 1, \qquad D = d, \qquad G = 2c\frac{\partial}{\partial x}$$
$$K = (c^2 - T_0)\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4} \tag{71}$$

By Eq. (11), the auxiliary system is found as

$$\left\{\frac{\partial^2}{\partial t^2} + \left(-2c\frac{\partial}{\partial x} + d\right)\frac{\partial}{\partial t} + (c^2 - T_0)\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right\}w_2(x, t)$$
$$= f_2(x, t), \qquad x \in (0, 1) \tag{72a}$$

$$w_2(0, t) = 0, \qquad \frac{\partial^2}{\partial x^2}w_2(0, t) = -\tau_{L2}(t)$$
$$w_2(1, t) = y_{R2}(t), \qquad \frac{\partial^2}{\partial x^2}w_2(1, t) = 0 \tag{72b}$$

Here, $\tau_{L2}(t)$ and $y_{R2}(t)$ are chosen for demonstrative purposes; other boundary disturbances can be assigned. Note that the boundary operators Γ_j and Γ_j^* are identical. In fact, the auxiliary system (72) is the original system (68) with a reversed transport speed $(-c)$; see Fig. 2b.

The eigenvalue problems associated with Eqs. (70) and (72) are described by

$$\left\{\lambda_k^2 + \lambda_k\left(2c\frac{\partial}{\partial x} + d\right) + (c^2 - T_0)\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right\}u_k(x) = 0$$
$$x \in (0, 1) \tag{73}$$

and

$$\left\{\bar{\lambda}_k^2 + \bar{\lambda}_k\left(-2c\frac{\partial}{\partial x} + d\right) + (c^2 - T_0)\frac{\partial^2}{\partial x^2} + \frac{\partial^4}{\partial x^4}\right\}v_k(x) = 0$$
$$x \in (0, 1) \tag{74}$$

where both problems have the same simplysupported boundary conditions. Owing to damping and gyroscopic forces in the translating beams, the eigenfunctions u_k and v_k are complex and nonorthogonal. It is from Eqs. (73) and (74) that

$$v_k(x) = u_{-k}(1 - x) = \bar{u}_k(1 - x), \qquad k = \pm 1, \pm 2, \ldots \tag{75}$$

Hence, the adjoint eigenfunctions $v_k(x)$ also represent the mode shapes of the original system (70). The exact eigensolutions of the translating beams can be obtained by a distributed transfer function method.[17]

The Green's function formula for the original system (70), according to Eq. (19), is

$$w_1(x, t) = \int_0^t \int_0^1 f_1(x, \tau)g(x, t - \tau)\,dx\,d\tau$$
$$+ \int_0^t \left\{-y_{L1}(\tau)\left[\frac{\partial^3}{\partial \xi^3} + (c^2 - T_0)\frac{\partial}{\partial \xi}\right]g(x, 0, t - \tau)\right.$$
$$\left. + \tau_{R1}(\tau)\frac{\partial}{\partial \xi}g(x, 1, t - \tau)\right\}d\tau + \int_0^1 \left\{\frac{\partial}{\partial t}g(x, \xi, t)a_1(x)\right.$$
$$\left. + g(x, \xi, t)\left[b_1(x) + \left(d + 2c\frac{\partial}{\partial x}\right)a_1(x)\right]\right\}dx \tag{76}$$

where $a_1(x)$ and $b_1(x)$ are the initial displacement and velocity of the moving beam (70). The Green's function in Eq. (76) is in the eigenfunction series

$$g(x, \xi, t) = \sum_{k=\pm 1}^{\pm\infty} \frac{1}{\sigma_k}e^{\lambda_k t}u_k(x)u_k(1 - \xi), \qquad x, \xi \in (0, 1) \tag{77}$$

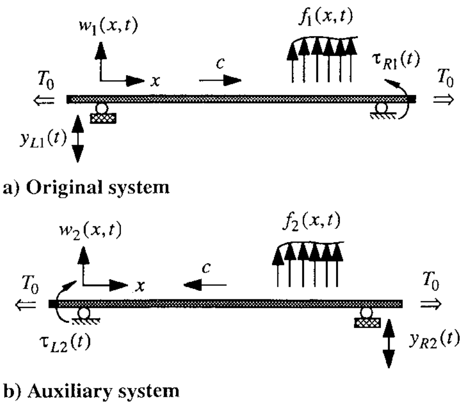

Fig. 2 Two translating beams.

where Eq. (75) has been used. Substituting Eq. (77) into Eq. (76) gives

$$w_1(x, t) = \sum_{k=\pm 1}^{\pm\infty} \frac{1}{\sigma_k}u_k(x)q_k(t) \tag{78}$$

where

$$q_k(t) = \int_0^t \int_0^1 u_k(1 - x)f_1(x, \tau)\exp[\lambda_k(t - \tau)]\,dx\,d\tau$$
$$+ \left[\frac{d^3}{dx^3}u_k(x) + (c^2 - T_0)\frac{\mathrm{d}}{\mathrm{d}x}u_k(x)\right]_{x=1}$$
$$\times \int_0^t y_{L1}(\tau)\exp[\lambda_k(t - \tau)]\,d\tau - \left[\frac{\mathrm{d}}{\mathrm{d}x}u_k(x)\right]_{x=0}$$
$$\times \int_0^t \tau_{R1}(\tau)\exp[\lambda_k(t - \tau)]\,d\tau + \int_0^1 u_k(1 - x)$$
$$\times \left[\left(\lambda_k + d + 2c\frac{\mathrm{d}}{\mathrm{d}x}\right)a_1(x) + b_1(x)\right]\mathrm{d}x\,e^{\lambda_k t} \tag{79a}$$

$$\sigma_k = \int_0^1 u_k(1 - x)\left[(2\lambda_k + d)u_k(x) + 2c\frac{\mathrm{d}}{\mathrm{d}x}u_k(x)\right]\mathrm{d}x \tag{79b}$$

By Theorem 4, the reciprocal relation for the translating beams is

$$\int_0^t \int_0^1 f_2(x, \tau)w_1(x, t - \tau)\,dx\,d\tau + \int_0^1 \left\{\frac{\partial}{\partial t}w_1(x, t)a_2(x)\right.$$
$$\left. + w_1(x, t)\left[b_2(x) + \left(d - 2c\frac{\partial}{\partial x}\right)a_2(x)\right]\right\}dx$$
$$+ \int_0^t \left\{-\tau_{L2}(\tau)\frac{\partial}{\partial x}w_1(0, t - \tau)\right.$$
$$\left. + y_{R2}(\tau)\left[\frac{\partial^3}{\partial x^3} + (c^2 - T_0)\frac{\partial}{\partial x}\right]w_1(1, t - \tau)\right\}d\tau$$
$$= \int_0^t \int_0^1 f_1(x, \tau)w_2(x, t - \tau)\,dx\,d\tau$$
$$+ \int_0^1 \left\{\frac{\partial}{\partial t}w_2(x, t)a_1(x) + w_2(x, t)\right.$$
$$\left. \times \left[b_1(x) + \left(d + 2c\frac{\partial}{\partial x}\right)a_1(x)\right]\right\}dx$$
$$+ \int_0^t \left\{-y_{L1}(\tau)\left[\frac{\partial^3}{\partial x^3} + (c^2 - T_0)\frac{\partial}{\partial x}\right]w_2(0, t - \tau)\right.$$
$$\left. + \tau_{R1}(\tau)\frac{\partial}{\partial x}w_2(1, t - \tau)\right\}d\tau \tag{80}$$

where $a_j(x)$ and $b_j(x)$ are the initial displacements and velocities of the beams. If the auxiliary system (72) is only subject to the impulse

$f_2(x, t) = \delta(x - \xi)\delta(t)$, its response is the Green's function of the original system (70), i.e., $w_2(x, t) = g(x, \xi, t)$. In this case, Eq. (80) reduces to the Green's function formula (76).

VII. Conclusions

A Green's function formula for a class of distributed dynamic systems is obtained through use of Laplace transform. This integral formula, Eq. (19), accounts for the non-self-adjoint effects of damping, gyroscopic, and circulatory forces, and explicitly describes the influence of arbitrary external, boundary, and initial disturbances by the impulse response of the adjoint system.

A relationship between the adjoint state–space eigenfunctions and the modes of vibration is established. This relationship makes it possible to represent the Green's function in a series of mode shape functions, i.e., the eigenfunctions associated with the original equations of motion. Additionally, the calculation of adjoint state–space eigenfunctions in a transient analysis is avoided, which implies potential savings in computation.

Theorem 2 provides a new way to calculate transfer function residues for non-self-adjoint systems, namely, to express residues by system eigensolutions. Often in an engineering problem, the transfer function of a complex distributed system can only be estimated numerically. Because the singularities of the transfer function at its poles can lead to large errors in computation, direct prediction of the residues by the transfer function itself is impractical. On the other hand, highly accurate solutions of eigenvalue problems (20) and (21) can be obtained by many well-developed techniques. Hence, Theorem 2 warrants precise and systematic estimation of transfer function residues for non-self-adjoint distributed systems.

The transient response and Green's function of a non-self-adjoint system are obtained in eigenfunction series, without the need to assume complete eigenfunctions. This interesting and important result may indicate that completeness of eigenfunctions is only a sufficient condition to guarantee a convergent-series solution. Although vibrating continua are analyzed here, similar modal expansion can be applied to other types of distributed dynamic systems in mathematical physics.

The response of a non-self-adjoint distributed system can be determined from that of the corresponding auxiliary system, which in many cases is the same as the original system. This reciprocal relation reduces to the Green's function formula for the original system when the auxiliary system is only under an impulsive load.

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References

[1]Butkoviskiy, A. G., *Structure Theory of Distributed Systems*, Wiley, New York, 1983.

[2]Courant, R., and Hilbert, D., *Methods of Mathematical Physics*, Wiley, New York, 1937.

[3]Morse, P. M., and Feshbach, H., *Methods of Theoretical Physics*, McGraw–Hill, New York, 1978.

[4]Roach, G. F., *Green's Functions*, 2nd ed., Cambridge Univ. Press, New York, 1982.

[5]Wickert, J. A., and Mote, C. D., Jr., "Classical Vibration of Axially Moving Continua," *Journal of Applied Mechanics*, Vol. 57, No. 4, 1990, pp. 738–744.

[6]Yang, B., and Mote, C. D., Jr., "Frequency-Domain Vibration Control of Distributed Gyroscopic Systems," *Journal of Dynamic Systems, Measurement and Control*, Vol. 113, No. 1, 1992, pp. 18–25.

[7]Meirovitch, L., and Silverberg, L. M., "Control of Non-Self-Adjoint Distributed Parameter Systems," *Journal of Optimization Theory and Applications*, Vol. 47, No. 1, 1985, pp. 77–90.

[8]Yang, B., "Eigenvalue Inclusion Principles for Distributed Gyroscopic Systems," *Journal of Applied Mechanics*, Vol. 59, No. 3, 1992, pp. 650–656.

[9]Yang, B., "Closed-Form Transient Response of Distributed Damped Systems, Part I: Modal Analysis and Green's Function Formula," *Journal of Applied Mechanics* (to be published).

[10]Love, A. E. H., *A Treatise on the Mathematical Theory of Elasticity*, Dover, New York, 1944.

[11]Gurtin, M. E., and Sternberg, E., "On the Linear Theory of Viscoelasticity," *Archive for Rational Mechanics and Analysis.*, Vol. 11, No. 4, 1962, pp. 291–356.

[12]Huseyin, K.,*Vibration and Stability of Multiple Parameter Systems*, Noordhoff International, Leyden, The Netherlands, 1978.

[13]Meirovitch, L., *Computational Methods in Structural Dynamics*, Sijthoff and Noordhoff, Alphenaan den Rijn, The Netherlands, 1980.

[14]Yang, B., "Modal Controllability and Observability of General Mechanical Systems," *Proceedings of the 1994 ASME Winter Annual Meeting*, Vol. DE-75, American Society of Mechanical Engineers, New York, 1994, pp. 363–370.

[15]Meirovitch, L., *Analytical Methods in Vibrations*, Macmillan, London, 1967.

[16]Mote, C. D., Jr., "Dynamic Stability of Axially Moving Materials," *Shock and Vibration Digest*, Vol. 4, No. 4, 1972, pp. 2–11.

[17]Yang, B., "Distributed Transfer Function Analysis of Complex Distributed Parameter Systems," *Journal of Applied Mechanics*, Vol. 61, No. 1, 1994, pp. 84–92.